A Simplified Description of Fuzzy TOPSIS

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Abstract—**A simplified description of Fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Situation) is presented. We have adapted the TOPSIS description from existing Fuzzy theory literature and distilled the bare minimum concepts required for understanding and applying TOPSIS. An example has been worked out to illustrate the application of TOPSIS for a multi-criteria group decision making scenario.**

Index Terms—**Evaluation method, Multi-criteria decision making, group decision making**

I. INTRODUCTION

Multi-criteria group decision making (MCGDM) problems are frequently encountered in practice. Several methods exist that can be applied to solve such problems. One example scenario of a MCGDM problem is where a group of two persons (A and B) intends to determine which laptop to buy based on certain criteria. Let's say they have evaluation criteria such as: price, screen size, memory and battery life. Each decision maker can have different importance for different criteria. For example, relative importance of criteria for A can be: price > battery life > memory > screen size. Here > means greater than. For B it can be some different ordering of criteria. Given this scenario, and a set of laptop alternatives, one wants to find which alternative best meets the group's criteria.

Fuzzy TOPSIS is a method that can help in objective and systematic evaluation of alternatives on multiple criteria. In Section II we provide a simplified description of required Fuzzy theory concepts. TOPSIS steps are explained in Section III, and an example scenario has been worked out in Section IV to illustrate the TOPSIS steps.

II. FUZZY THEORY DETAILS

A detailed introduction and treatment of the fuzzy set theory is available in [1], [2]. The definitions of fuzzy concepts that are relevant for understanding of TOPSIS have been adapted from these sources. These definitions are presented as follows.

Definition 1: A fuzzy set $\tilde{a}$ in a universe of discourse X is characterized by a membership function $\mu_{\tilde{a}}(x)$ that maps each element x in X to a real number in the interval [0, 1]. The function value $\mu_{\tilde{a}}(x)$ is termed the grade of membership of x in $\tilde{a}$. The nearer the value of $\mu_{\tilde{a}}(x)$ to unity, the higher the grade of membership of x in $\tilde{a}$.

Definition 2: A triangular fuzzy number is represented as a triplet $\tilde{a} = (a_1, a_2, a_3)$. The membership function $\mu_{\tilde{a}}(x)$

TABLE I
FUZZY RATINGS FOR LINGUISTIC VARIABLES

Fuzzy number	Alternative Assessment	QA Weights
(1,1,3)	Very Poor (VP)	Very Low (VL)
(1,3,5)	Poor (P)	Low (L)
(3,5,7)	Fair (F)	Medium (M)
(5,7,9)	Good (G)	High (H)
(7,9,9)	Very Good (VG)	Very High (VH)

of triangular fuzzy number $\tilde{a}$ is given as:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3 \\ 0 & Otherwise \end{cases} \quad (1)$$

where a_1, a_2, a_3 are real numbers and $a_1 < a_2 < a_3$. The value of x at a_2 gives the maximal grade of $\mu_{\tilde{a}}(x)$, i.e., $\mu_{\tilde{a}}(x) = 1$; it is the most probable value of the evaluation data. The value of x at a_1 gives the minimal grade of $\mu_{\tilde{a}}(x)$, i.e., $\mu_{\tilde{a}}(x) = 0$; it is the least probable value of the evaluation data. Constants a_1 and a_3 are the lower and upper bounds of the available area for the evaluation data. These constants reflect the fuzziness of the evaluation data. The narrower the interval $[a_1, a_3]$ the lower the fuzziness of the evaluation data.

A. The distance between fuzzy triangular numbers

Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. The distance between them is given using the vertex method by:

$$d(\tilde{a}, \tilde{b}) = \sqrt{\frac{1}{3}[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2]} \quad (2)$$

B. Linguistic variables

In fuzzy set theory, conversion scales are applied to transform the linguistic terms into fuzzy numbers. In this paper, we will apply a scale of 1 to 9 for rating the criteria and the alternatives. Following are the linguistic variables and fuzzy ratings for the alternatives and the criteria:

The values for the triangular fuzzy numbers that we have chosen for the linguistic variables take into consideration the fuzziness and the distance among the variables (please see equations 1 and 2). The intervals are chosen so as to have a uniform representation from 1 to 9 for the fuzzy triangular numbers used for the five linguistic ratings. For instance, one can also choose (4,5,6) instead of (1,1,3) to represent *Very Low* if one wishes so, however in that case, the "1 to 9" ratings would begin from 4 instead of 1. Normalization step takes care

of such shifting of the rating scale. The common practice in literature is to start the ratings scales from 1.

III. Fuzzy TOPSIS

The technique called fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Situation) can be used to evaluate multiple alternatives against the selected criteria. In the TOPSIS approach an alternative that is nearest to the Fuzzy Positive Ideal Solution (FPIS) and farthest from the Fuzzy Negative Ideal Solution (FNIS) is chosen as optimal. An FPIS is composed of the best performance values for each alternative whereas the FNIS consists of the worst performance values. A detailed description and treatment of TOPSIS is discussed by [1], [3] and we have adapted the relevant steps of fuzzy TOPSIS as presented below.

Let's say the decision group has K members. If the fuzzy rating and importance weight of the kth decision maker, about the ith alternative on jth criterion, are:
$\tilde{x}_{ij}^k = (a_{ij}^k, b_{ij}^k, c_{ij}^k)$ and $\tilde{w}_j^k = (w_{j1}^k, w_{j2}^k, w_{j3}^k)$ respectively, where $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, then the aggregated fuzzy ratings $\tilde{x}_{ij}$ of alternatives (i) with respect to each criterion (j) are given by $\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ such that:

$$a_{ij} = \min_k\{a_{ij}^k\}, \quad b_{ij} = \frac{1}{K}\sum_{k=1}^{K} b_{ij}^k, \quad c_{ij} = \max_k\{c_{ij}^k\} \quad (3)$$

The aggregated fuzzy weights ($\tilde{w}_{ij}$) of each criterion are calculated as $\tilde{w}_j^k = (w_{j1}, w_{j2}, w_{j3})$ where:

$$w_{j1} = \min_k\{w_{jk1}\}, \quad w_{j2} = \frac{1}{K}\sum_{k=1}^{K} w_{jk2}, \quad w_{j3} = \max_k\{w_{jk3}\} \quad (4)$$

A fuzzy multicriteria Group Decision Making (GDM) problem which can be concisely expressed in matrix format as:

$$\tilde{D} = \begin{array}{c} \\ A_1 \\ A_2 \\ \\ A_m \end{array} \begin{array}{c} C_1 \quad C_2 \quad\quad C_n \\ \left(\begin{array}{cccc} \tilde{x}_{11} & \tilde{x}_{12} & \ldots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \ldots & \tilde{x}_{2n} \\ \ldots & \ldots & \tilde{x}_{ij} & \ldots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \ldots & \tilde{x}_{mn} \end{array} \right) \end{array} \quad (5)$$

$$\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n) \quad (6)$$

where $\tilde{x}_{ij}, \forall i, j$ and $\tilde{w}_j, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ are linguistic variables which can be described by triangular fuzzy numbers, $\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ and $\tilde{w}_j = (w_{j1}, w_{j2}, w_{j3})$. To keep the normalization formula simple, the linear scale transformation is used to transform various criteria scales into a comparable scale. Thus, we have the normalized fuzzy decision matrix as:

$$\tilde{R} = [\tilde{r}_{ij}]_{m \times n}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n \quad (7)$$

where:

$$\left. \begin{array}{l} \tilde{r}_{ij} = \left(\dfrac{a_{ij}}{c_j^*}, \dfrac{b_{ij}}{c_j^*}, \dfrac{c_{ij}}{c_j^*} \right) \quad \text{and} \\[2ex] c_j^* = \max_i c_{ij} \quad \text{(benefit criteria)} \end{array} \right\} \quad (8)$$

$$\left. \begin{array}{l} \tilde{r}_{ij} = \left(\dfrac{a_j^-}{c_{ij}}, \dfrac{a_j^-}{b_{ij}}, \dfrac{a_j^-}{a_{ij}} \right) \quad \text{and} \\[2ex] a_j^- = \min_i a_{ij} \quad \text{(cost criteria)} \end{array} \right\} \quad (9)$$

The above normalization method preserves the property that the ranges of normalized triangular fuzzy numbers belong to $[0, 1]$.

The weighted normalized fuzzy decision matrix $\tilde{V}$ is computed by multiplying the weights ($\tilde{w}_j$) of evaluation criteria with the normalized fuzzy decision matrix $\tilde{r}_{ij}$ as:

$$\tilde{V} = [\tilde{v}_{ij}]_{m \times n}, \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n \\ \text{where} \quad \tilde{v}_{ij} = \tilde{r}_{ij}(\cdot)\tilde{w}_j \quad (10)$$

The FPIS and FNIS of the alternatives are defined as follows:

$$A^* = (\tilde{v}_1^*, \tilde{v}_2^*, \ldots, \tilde{v}_n^*) \quad \text{where} \\ \tilde{v}_j^* = \max_i\{v_{ij3}\}, i = 1, 2 \ldots, m; j = 1, 2, \ldots, n \quad (11)$$

$$A^- = (\tilde{v}_1^-, \tilde{v}_2^-, \ldots, \tilde{v}_n^-) \quad \text{where} \\ \tilde{v}_j^- = \min_i\{v_{ij1}\}, i = 1, 2 \ldots, m; j = 1, 2, \ldots, n \quad (12)$$

The distance (d_i^* and d_i^-) of each weighted alternative $i = 1, 2, \ldots, m$ from the FPIS and the FNIS is computed as follows:

$$d_i^* = \sum_{j=1}^{n} d_v(\tilde{v}_{ij}, \tilde{v}_j^*), \quad i = 1, 2, \ldots, m \quad (13)$$

$$d_i^- = \sum_{j=1}^{n} d_v(\tilde{v}_{ij}, \tilde{v}_j^-), \quad i = 1, 2, \ldots, m \quad (14)$$

where $d_v(\tilde{a}, \tilde{b})$ is the distance measurement between two fuzzy number $\tilde{a}$ and $\tilde{b}$.

The closeness coefficient CC_i represents the distances to fuzzy positive ideal solution, A^*, and the fuzzy negative ideal solution, A^- simultaneously. The closeness coefficient of each alternative is calculated as:

$$CC_i = \frac{d_i^-}{d_i^- + d_i^*}, \quad i = 1, 2 \ldots, m \quad (15)$$

The alternative with highest closeness coefficient represents the best alternative and is closest to the FPIS and farthest from

TABLE II
CRITERIA WEIGHTAGE BY DECISION MAKERS

	C1	C2	C3	C4
DM_1	H (5, 7, 9)	VH (7, 9, 9)	VH (7, 9, 9)	M (3, 5, 7)
DM_2	M (3, 5, 7)	H (5, 7, 9)	H (5, 7, 9)	L (1, 3, 5)
DM_3	M (3, 5, 7)	H (5, 7, 9)	H (5, 7, 9)	L (1, 3, 5)

TABLE III
ALTERNATIVES RATINGS BY DECISION MAKERS

Criteria	A1			A2		
	DM1	DM2	DM3	DM1	DM2	DM3
C1	F	F	F	G	G	F
C2	VG	VG	VG	G	VG	G
C3	P	F	P	P	P	P
C4	F	F	P	P	P	F

TABLE VI
WEIGHTED NORMALIZED FUZZY DECISION MATRIX

Criteria	A1	A2
C1	(1.286, 3.400, 9.000)	(1.000, 2.684, 9.000)
C2	(3.889, 7.667, 9.000)	(2.778, 6.531, 9.000)
C3	(0.714, 4.016, 9.000)	(0.714, 3.286, 6.429)
C4	(0.143, 2.270, 7.000)	(0.143, 1.921, 7.000)

TABLE VII
DISTANCES $d_v(A_i, A^*)$ AND $d_v(A_i, A^-)$ FROM FPIS AND FNIS FOR ALTERNATIVES

Criteria	FPIS(A_1)	FPIS(A_2)	FNIS(A_1)	FNIS(A_2)
C1	5.503	5.884	4.824	4.72
C2	3.049	3.864	4.613	4.195
C3	5.582	5.997	5.149	3.617
C4	4.809	4.926	4.145	4.089

the FNIS. In summary, the TOPSIS steps that we use are as below:

i. Aggregate the weight of criteria to get the aggregated fuzzy weight $\tilde{w}_j$ of criterion C_j (using equation 4) and pool the decision makers' ratings to get the aggregated fuzzy rating $\tilde{x}_{ij}$ of alternative A_i under criterion C_j (using equation 3).

ii. Construct the fuzzy decision matrix and the normalized fuzzy decision matrix (equations 7, 8 and 9).

iii. Construct the weighted normalized fuzzy decision matrix (equation 10).

iv. Determine FPIS and FNIS and calculate the distance of each alternative from FPIS and FNIS, respectively (equations 11, 12, 13 and 14).

v. Calculate the closeness coefficient of each alternative and rank the alternatives (equation 15).

IV. NUMERICAL ILLUSTRATION

Below we describe the application of our TOPSIS for a scenario where there are 3 decision makers, 4 evaluation criteria C1 – C4, and rating scale is as shown in Table I.

In this example, a team of three decision makers D_1, D_2 and D_3 is formed to evaluate the two alternatives, A_1 and A_2,

for picking the optimal one. Key input from decision makers is typically to identify the proper weightage to various criteria.

The team provided linguistic weightage for the criteria in Table II, and assessment for three alternatives on each of the criteria item is presented in Table III.

Results of various TOPSIS calculation steps are shown in Tables IV – VII. Closeness coefficients, CC_i, of the two alternatives A_1 and A_2 come out to be 0.497 and 0.445 respectively. Hence the ranking order for the alternatives is $A_1 > A_2$, that is, A_1 is the best choice considering the given criteria. The closeness coefficient scores for alternatives are numeric values and can be further utilized to indicate the degree of inferiority or superiority of the alternatives w.r.t each other.

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TABLE IV
AGGREGATE FUZZY DECISION MATRIX

Criteria	A1	A2
C1	(3.000, 5.000, 7.000)	(3.000, 6.333, 9.000)
C2	(7.000, 9.000, 9.000)	(5.000, 7.667, 9.000)
C3	(1.000, 3.667, 7.000)	(1.000, 3.000, 5.000)
C4	(1.000, 4.333, 7.000)	(1.000, 3.667, 7.000)

TABLE V
NORMALIZED AGGREGATE FUZZY DECISION MATRIX

Criteria	A1	A2
C1	(0.429, 0.600, 1.000)	(0.333, 0.474, 1.000)
C2	(0.778, 1.000, 1.000)	(0.556, 0.852, 1.000)
C3	(0.143, 0.524, 1.000)	(0.143, 0.429, 0.714)
C4	(0.143, 0.619, 1.000)	(0.143, 0.524, 1.000)